A Substitution Theorem for Graceful Trees and its Applications*

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Abstract

A *graceful labeling* of a graph $G = (V, E)$ assigns $|V|$ distinct integers from the set $\{0, \ldots, |E|\}$ to the vertices of G so that the absolute values of their differences on the $|E|$ edges of G constitute the set $\{1, \ldots, |E|\}$. A graph is *graceful* if it admits a graceful labeling. The forty-year old *Graceful Tree Conjecture*, due to Ringel and Kotzig, states that every tree is graceful.

We prove a *Substitution Theorem* for graceful trees, which enables the construction of a larger graceful tree through combining smaller and not necessarily identical graceful trees. We present applications of the Substitution Theorem, which generalize earlier constructions combining smaller trees.

Key Words: graceful tree, graceful labeling, gracefully consistent trees, substitution theorem.

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1 Introduction

A *labeling* of a graph $G = (V, E)$ is an assignment $\theta : V \to \{0, \ldots, |E|\}$ of *labels* to the vertices of G that induces for each edge uv an *edge label* depending on the labels $\theta(u)$ and $\theta(v)$ (cf. [4]). A *graceful labeling* [14] of $G = (V, E)$ is an injection from the vertices of G to the set $\{0, \ldots, |E|\}$ such that when each edge uv is assigned the label $|\theta(u) - \theta(v)|$, the resulting edge labels are all distinct; so, $\{|\theta(u) - \theta(v)| \mid uv \in E\} = \{1, \ldots, |E|\}$. A *graceful* graph is one admitting a graceful labeling. When the graph G is a tree, graceful labeling implies that θ is a bijection. The long-standing *Ringel-Kotzig Conjecture* [13], also known as the *Graceful Tree Conjecture*, states that all trees are graceful. Not too many classes of trees are yet known to be graceful — see, e.g., [1, 2, 6, 12, 13].

One successful approach toward extending the class of trees known to be graceful builds on combining together or modifying trees already known to be graceful, henceforth called the *constituent trees*, to produce a larger graceful tree, henceforth called the *constructed tree* (cf. [8]). This approach has been taken, for example, in the following works:

Koh *et al.* [7, 9] and Rosa and Širáň [15] connect together the constituent trees by attaching their roots in certain ways; they prove that the resulting tree is also graceful. Lladó and López [10] extend the constructions in [7, 9, 15] so that they apply to the case where each of the constituent trees is *bigraceful*; the constructed tree is then bigraceful. A bigraceful labeling [11] is strictly weaker than a graceful labeling; the essential deviation is that the labeling is no longer an injection.

Burzio and Ferrarese [3] prove that the tree obtained by subdividing every edge in a graceful tree is also graceful. Furthermore, they provide ways to attach the constituent trees to vertices of a given *host* tree. Lladó and López [10] extend some of the constructions in [3] to the case of bigraceful labeling.

In this article we present and explore the *Substitution Theorem* for graceful trees (Theorem 2.3), a combinatorial tool that enables extending previous results on combining a family of copies of a graceful tree into a larger graceful tree. Its contribution lies in relaxing the requirement that the constituent trees be identical; instead it allows arbitrary families of graceful trees to be combined as long as these trees satisfy a certain combinatorial property.

The Substitution Theorem applies to families of gracefully labeled trees that are pairwise *gracefully consistent*; a pair of gracefully labeled trees is gracefully consistent if there exists an integer k such that the graceful labeling of each tree assigns to each pair of adjacent vertices a label larger than k and a label at most equal to k. Such labelings will be called *strongly graceful labelings*, and k will be called the *strength* of the labeling. A family of trees that admit strongly graceful labelings with the same strength will be called a *gracefully consistent family*; each tree in the family has a designated vertex, called the *root*. Through an application of the Substitution Theorem to a gracefully consistent family, we obtain the following results:

(1) We present the *Extended Garland Construction*, where the roots of the constituent trees from a gracefully consistent family are connected to a new distinct vertex (Theorem 3.2). This extends the class of graceful trees resulting from the original *Garland Construction* in [7].

Lladó and López [10, Lemma 2.2] provide a generalization of the Garland Construction where the roots of the constituent trees are identified with the leaves of an arbitrary tree. In their generalization, the constituent trees are only bigraceful, as also is the constructed tree.

(2) We present the *Extended Attachment Construction*, where the roots of the constituent trees from a gracefully consistent family are unified into a single vertex (Theorem 3.4). This extends the class of graceful trees resulting from the original *Attachment Construction* in [9].

Lladó and López [10, Lemma 2.1] show that the Attachment Construction can be applied to two arbitrary bigraceful trees to construct a new bigraceful tree; if the two trees are strongly graceful (but not necessarily gracefully consistent), the constructed tree is strongly graceful. The root (vertex labeled with 0) of the constructed tree is *necessarily* different than the vertex unifying the roots of the two constituent trees. To extend to an arbitrary number of constituent trees, the new root must be identified with the root of the third constituent tree, and so on. In contrast, the roots of all constituent trees in the construction from Theorem 3.4 are unified together. Hence, the two results are incomparable.

Rosa and Širáň [15, Lemma 2] show that the Attachment Construction can be applied to two arbitrary strongly graceful trees to construct a new graceful tree. Much in the same way as [10, Lemma 2.1], this result is incomparable to Theorem 3.4.

(3) We present the *Extended* Δ*-Construction*, where the roots of the constituent trees from a gracefully consistent family are unified with the vertices of some fixed but arbitrary tree, called the *host* (Theorem 3.6). This extends the class of graceful trees resulting from the original Δ*-Construction* in [9].

The construction in [10, Lemma 2.1] is the special case of the Δ-Construction where the host tree is a single edge. However, this construction assumes that the two arbitrary constituent trees are bigraceful and the constructed tree is bigraceful. In [10, discussion following Lemma 2.1], the authors note that when their construction is applied to two strongly graceful (but not necessarily gracefully consistent) trees, the constructed tree is strongly graceful. Since there is no assumption on the graceful consistency of the constituent trees in [10, Lemma 2.1], this result is strictly stronger than Theorem 3.6, but it only applies to the special case where the host tree is a single edge.

The construction in [10, Lemma 2.2] generalizes the Δ-Construction where the roots of the constituent trees are unified with some leaves of the host tree. Furthermore, it is assumed in [10, Lemma 2.2] only that all constituent trees have the same number of edges, while Theorem 3.6 assumes that they come from a gracefully consistent family. However, [10, Lemma 2.2] applies when the constituent trees are bigraceful and yields a bigraceful tree.

We also present the *Generalized Extended* Δ*-Construction*, which parallels the *Generalized* Δ*-Construction* in [3] by allowing certain edges to be moved around in a tree constructed via the Extended Δ-Construction while preserving its gracefulness (Theorem 3.7).

(4) We present the *Extended* Δ_{+1}*-Construction*, which resembles the Extended Δ-Construction; however, it allows for one vertex of the host tree not to be unified with a root (Theorem 3.9). This extends the class of graceful trees resulting from the original Δ_{+1}*-Construction* in [3].

We also present the *Generalized Extended* Δ_{+1}*-Construction*, which parallels the *Generalized* Δ_{+1}*-Construction* by allowing certain edges to be moved around in a tree constructed via the Extended Δ_{+1}-Construction while preserving its gracefulness (Theorem 3.10).

2 The Substitution Theorem

We will focus on graceful labelings for *trees*, where a graceful labeling θ for a tree $T = (V, E)$ is a bijection from V to $\{0, \ldots, |E|\}$. For a graceful labeling θ, the vertex assigned the value 0 will be called the 0-*vertex* of θ and denoted as 0_θ; thus, $\theta(0_\theta) = 0$. In what follows, we will use a triple $\langle T, \theta, w\rangle$ to simultaneously refer to a gracefully labeled tree T, its graceful labeling θ, and a distinguished vertex $w \in V(T)$, which we will call the *root*. In general, the root is a single vertex that satisfies some property specific to each particular construction. By $\mathsf{dist}(u, v)$ we will denote the distance between u and v in a tree.

2.1 Strongly Graceful Labeling

The definition of strongly graceful labeling is due to Rosa [14], who called it α-*valuation*. For a gracefully labeled tree $\langle T, \theta, w\rangle$, say that θ is a *strongly graceful labeling of T with strength $k \in \mathbb{N}$* if for every edge $uv \in E(T)$, either $\theta(u) \le k < \theta(v)$ or $\theta(v) \le k < \theta(u)$. In such case, say that $\langle T, \theta, w\rangle$ is *strongly gracefully labeled with strength $k \in \mathbb{N}$*. Define now the sets

$$\mathsf{EvenLabels}(\langle T, \theta, w\rangle) = \{\theta(v) \mid v \in V(T) \text{ and } \mathsf{dist}(v, w) \text{ is even}\}$$

and

$$\mathsf{OddLabels}(\langle T, \theta, w\rangle) = \{\theta(v) \mid v \in V(T) \text{ and } \mathsf{dist}(v, w) \text{ is odd}\}\,.$$

The following claim is a direct consequence of the bipartiteness of trees:

Lemma 2.1 *Consider a strongly gracefully labeled tree $\langle T, \theta, w\rangle$ with strength $k \in \mathbb{N}$. Then:*

(1) If $\theta(w) \le k$, then
$\mathsf{EvenLabels}(\langle T, \theta, w\rangle) = \{0, \ldots, k\}$ *and* $\mathsf{OddLabels}(\langle T, \theta, w\rangle) = \{k + 1, \ldots, |V(T)| - 1\}$.

(2) If $\theta(w) > k$, then
$\mathsf{OddLabels}(\langle T, \theta, w\rangle) = \{0, \ldots, k\}$ *and* $\mathsf{EvenLabels}(\langle T, \theta, w\rangle) = \{k + 1, \ldots, |V(T)| - 1\}$.

2.2 Gracefully Consistent Trees

Say that two gracefully labeled trees $\langle T_1, \theta_1, w_1\rangle$ and $\langle T_2, \theta_2, w_2\rangle$ with $|V(T_1)| = |V(T_2)|$ are gracefully consistent if either of the following conditions holds:

(1) The gracefully labeled trees $\langle T_1, \theta_1, w_1\rangle$ and $\langle T_2, \theta_2, w_2\rangle$ are identical.

(2) The labelings θ_1 and θ_2 are strongly graceful with the same strength, and $\theta_1(w_1) = \theta_2(w_2)$.

Say that a family of gracefully labeled trees is gracefully consistent if the trees in the family are pairwise gracefully consistent. Observe that for any pair of gracefully consistent trees $\langle T_1, \theta_1, w_1\rangle$ and $\langle T_2, \theta_2, w_2\rangle$, $\mathsf{EvenLabels}(\langle T_1, \theta_1, w_1\rangle) = \mathsf{EvenLabels}(\langle T_2, \theta_2, w_2\rangle)$ and $\mathsf{OddLabels}(\langle T_1, \theta_1, w_1\rangle) = \mathsf{OddLabels}(\langle T_2, \theta_2, w_2\rangle)$. This property is illustrated in Figure 1.

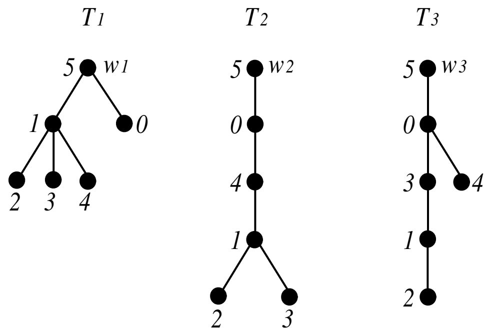


Figure 1: The gracefully consistent family $\{\langle T_1, \theta_1, w_1\rangle, \langle T_2, \theta_2, w_2\rangle, \langle T_3, \theta_3, w_3\rangle\}$.

2.3 Relabeling Function

We define:

Definition 2.1 (Relabeling Function) *Consider a gracefully labeled tree $\langle T, \theta, w\rangle$, and a triple of integers $\langle c, e, o\rangle \in \mathbb{Z}^3$. Define the* relabeling function $\mathcal{R}^{\langle T,\theta,w\rangle}_{\langle c,e,o\rangle} : V(T) \to \mathbb{Z}$ *with*

$$\mathcal{R}^{\langle T,\theta,w\rangle}_{\langle c,e,o\rangle}(v) \quad = \quad \begin{cases} c(\theta(v) + e)\,, & \text{if } \mathsf{dist}(v, w) \text{ is even} \\ c(\theta(v) + o)\,, & \text{if } \mathsf{dist}(v, w) \text{ is odd} \end{cases}$$

The triple $\langle c, e, o\rangle$ and the root of the tree T in the definition of the relabeling function depend on each specific construction. Roughly speaking, the integers e and o correspond to offsets applied, respectively, to the labels of vertices at even and odd distance from the root, while the integer c corresponds to a multiplicative factor applied to the labels of all vertices. We observe:

Lemma 2.2 *Consider a strongly gracefully labeled tree $\langle T, \theta, w\rangle$ with strength $k \in \mathbb{N}$. Then, for all edges $uv \in E(T)$,*

$$\left|\mathcal{R}^{\langle T,\theta,w\rangle}_{\langle c,e,o\rangle}(u) - \mathcal{R}^{\langle T,\theta,w\rangle}_{\langle c,e,o\rangle}(v)\right| = \begin{cases} |c|\,||\theta(u) - \theta(v)| - e + o|\,, & \text{if } \theta(w) \le k \\ |c|\,||\theta(u) - \theta(v)| + e - o|\,, & \text{if } \theta(w) > k \end{cases},$$

so that

$$\left\{\left|\mathcal{R}^{\langle T,\theta,w\rangle}_{\langle c,e,o\rangle}(u) - \mathcal{R}^{\langle T,\theta,w\rangle}_{\langle c,e,o\rangle}(v)\right| \mid uv \in E(T)\right\} = \begin{cases} \{|c|\,|\delta - e + o| \mid 1 \le \delta \le |V(T)| - 1\}\,, & \text{if } \theta(w) \le k \\ \{|c|\,|\delta + e - o| \mid 1 \le \delta \le |V(T)| - 1\}\,, & \text{if } \theta(w) > k \end{cases}.$$

The claim follows immediately from Definition 2.1 and Lemma 2.1.

2.4 The Theorem

We are now ready to state and prove the Substitution Theorem:

Theorem 2.3 *Consider any pair of gracefully consistent trees $\langle T_1, \theta_1, w_1\rangle$ and $\langle T_2, \theta_2, w_2\rangle$. Then, for all triples of integers $\langle c, e, o\rangle \in \mathbb{Z}^3$,*

(1) $\mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(w_1) = \mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(w_2)$;

(2) $\left\{\mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(v) \mid v \in V(T_1)\right\} = \left\{\mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(v) \mid v \in V(T_2)\right\}$;

(3) $\left\{\left|\mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(u) - \mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(v)\right| \mid uv \in E(T_1)\right\} = \left\{\left|\mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(u) - \mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(v)\right| \mid uv \in E(T_2)\right\}$.

Proof: Assume that the labelings θ_1 and θ_2 are strongly graceful with the same strength $k \in \mathbb{N}$, and $\theta_1(w_1) = \theta_2(w_2)$.

For *(1)*, since $\mathsf{dist}(w_1, w_1) = \mathsf{dist}(w_2, w_2) = 0$, the definition of relabeling function implies that $\mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(w_1) = c(\theta_1(w_1)+e)$ and $\mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(w_2) = c(\theta_2(w_2)+e)$. By assumption $\theta_1(w_1) = \theta_2(w_2)$. It follows that $\mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(w_1) = \mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(w_2)$.

For *(2)*, the definition of relabeling function and an earlier observation imply that

$$\begin{aligned}
&\left\{\mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(v) \mid v \in V(T_1)\right\} \\
&= \{c(\theta_1(v)+e) \mid v \in V(T_1) \text{ and } \mathsf{dist}(v, w_1) \text{ is even}\} \cup \{c(\theta_1(v)+o) \mid v \in V(T_1) \text{ and } \mathsf{dist}(v, w_1) \text{ is odd}\} \\
&= \{c(\theta_2(v)+e) \mid v \in V(T_2) \text{ and } \mathsf{dist}(v, w_2) \text{ is even}\} \cup \{c(\theta_2(v)+o) \mid v \in V(T_2) \text{ and } \mathsf{dist}(v, w_2) \text{ is odd}\} \\
&= \left\{\mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(v) \mid v \in V(T_2)\right\}.
\end{aligned}$$

For *(3)*, Lemma 2.2 implies that

$$\begin{aligned}
&\left\{\left|\mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(u) - \mathcal{R}^{\langle T_1,\theta_1,w_1\rangle}_{\langle c,e,o\rangle}(v)\right| \mid uv \in E(T_1)\right\} \\
&= \begin{cases} \{|c|\,|\delta - e + o| \mid 1 \le \delta \le |V(T_1)| - 1\}, & \text{if } \theta_1(w_1) \le k \\ \{|c|\,|\delta + e - o| \mid 1 \le \delta \le |V(T_1)| - 1\}, & \text{if } \theta_1(w_1) > k \end{cases} \\
&= \begin{cases} \{|c|\,|\delta - e + o| \mid 1 \le \delta \le |V(T_2)| - 1\}, & \text{if } \theta_2(w_2) \le k \\ \{|c|\,|\delta + e - o| \mid 1 \le \delta \le |V(T_2)| - 1\}, & \text{if } \theta_2(w_2) > k \end{cases} \\
&= \left\{\left|\mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(u) - \mathcal{R}^{\langle T_2,\theta_2,w_2\rangle}_{\langle c,e,o\rangle}(v)\right| \mid uv \in E(T_2)\right\}.
\end{aligned}$$

Since the three conditions also hold (trivially) when $\langle T_1, \theta_1, w_1\rangle \equiv \langle T_2, \theta_2, w_2\rangle$, the claim follows. ■

The Substitution Theorem implies that applying the same relabeling function on any of a pair of gracefully consistent trees produces the same sets of vertex and edge labels; moreover, the roots of the two relabeled trees have the same label.

3 Applications

Unless otherwise stated, the family $\mathcal{S} = \{\langle T_1, \theta_1, w_1\rangle, \ldots, \langle T_h, \theta_h, w_h\rangle\}$ of gracefully labeled trees is employed in all constructions.

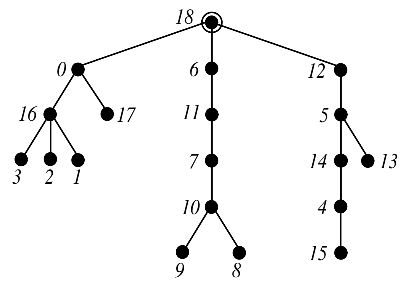


Figure 2: The tree $\textsc{garland}(\mathcal{S})$, when $\mathcal{S}$ consists of the trees $\langle T_1, \theta_1, w_1\rangle$, $\langle T_2, \theta_2, w_2\rangle$, and $\langle T_3, \theta_3, w_3\rangle$ taken from Figure 1. The special vertex of the constructed tree is circled.

3.1 The Garland Construction

Denote by $\textsc{garland}(\mathcal{S})$ the tree constructed by connecting a distinguished vertex r to the roots of all the trees in $\mathcal{S}$; we call r the *special vertex* of the constructed tree. The original construction is due to Koh *et al.* [7]; Goldenberg [5] calls it the *Garland Construction.* Koh *et al.* [7] prove:

Proposition 3.1 *Consider a gracefully labeled tree $\langle T, \theta, w\rangle$ with $\theta(w) = |V(T)| - 1$. Let $\mathcal{S}$ consist of h copies of $\langle T, \theta, w\rangle$. Then, the labeling*

$$\theta^*(v) = \begin{cases} h|V(T)|\,, & \text{if } v = r \\ -(\theta_i(v) + 1 - i|V(T_i)|)\,, & \text{if } v \in V(T_i) \text{ and } \mathsf{dist}(v, w_i) \text{ is even} \\ -(\theta_i(v) + 1 - (h+1-i)|V(T_i)|)\,, & \text{if } v \in V(T_i) \text{ and } \mathsf{dist}(v, w_i) \text{ is odd} \end{cases}$$

is a graceful labeling for the tree $\textsc{garland}(\mathcal{S})$.

Note that the relabeling function used on copy $\langle T_i, \theta_i, w_i\rangle$ is $\mathcal{R}^{\langle T_i,\theta_i,w_i\rangle}_{\langle -1, 1-i|V(T_i)|, 1-(h+1-i)|V(T_i)|\rangle}$.

The Extended Garland Construction requires that $\mathcal{S}$ be a gracefully consistent family; it returns a graceful labeling θ^* for the tree $\textsc{garland}(\mathcal{S})$ as follows:

$$\theta^*(v) = \begin{cases} h|V(T_1)| \quad (= h|V(T_2)| = \ldots = h|V(T_h)|)\,, & \text{if } v = r \\ \mathcal{R}^{\langle T_i,\theta_i,w_i\rangle}_{\langle -1, 1-i|V(T_i)|, 1-(h+1-i)|V(T_i)|\rangle}(v)\,, & \text{if } v \in V(T_i) \end{cases}.$$

Figure 2 provides an illustration for the Extended Garland Construction. We show:

Theorem 3.2 *Consider a gracefully consistent family $\mathcal{S}$ with $\theta_i(w_i) = |V(T_i)| - 1$, $1 \leq i \leq h$. Then, the Extended Garland Construction provides a graceful labeling θ^* for the tree $\textsc{garland}(\mathcal{S})$.*

Proof: Consider the family $\overline{\mathcal{S}} = \left\{\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle \mid \langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle \equiv \langle T_h, \theta_h, w_h\rangle,\ 1 \leq i \leq h\right\}$. By Proposition 3.1, the tree $\textsc{garland}(\overline{\mathcal{S}})$ is graceful. Recall that the labeling $\overline{\theta^*}$ for the tree $\textsc{garland}(\overline{\mathcal{S}})$ is obtained by relabeling every tree $\overline{T_i}$, $1 \leq i \leq h$ using the function $\mathcal{R}^{\langle \overline{T_i},\overline{\theta_i},\overline{w_i}\rangle}_{\langle -1, 1-i|V(\overline{T_i})|, 1-(h+1-i)|V(\overline{T_i})|\rangle}$; the labeling θ^* for the tree $\textsc{garland}(\mathcal{S})$ is obtained by relabeling every tree T_i, $1 \leq i \leq h$ using the function $\mathcal{R}^{\langle T_i,\theta_i,w_i\rangle}_{\langle -1, 1-i|V(T_i)|, 1-(h+1-i)|V(T_i)|\rangle}$.

Since $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle \equiv \langle T_h, \theta_h, w_h \rangle$ and $\langle T_i, \theta_i, w_i \rangle$ are gracefully consistent, $|V(\overline{T_i})| = |V(T_i)|$; thus, for each i, $1 \leq i \leq h$, $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle$ and $\langle T_i, \theta_i, w_i \rangle$ are relabeled using a relabeling function with the same triple of integers $\langle c, e, o \rangle$. Hence, by the Substitution Theorem, we get that for each i, $1 \leq i \leq h$: *(i)* $\overline{\theta^*}$ and θ^* assign the same labels to the roots of $\overline{T_i}$ and T_i, respectively; *(ii)* $\overline{\theta^*}$ and θ^* assign the same vertex and edge labels to the trees $\overline{T_i}$ and T_i, respectively. In addition, $\overline{\theta^*}$ and θ^* assign the same label to the special vertices $\overline{r}$ and r of the constructed trees GARLAND($\overline{\mathcal{S}}$) and GARLAND($\mathcal{S}$), respectively. Hence, the labels of the edges that are adjacent to $\overline{r}$ and r are also the same under the two labelings.

In conclusion, the two labelings $\overline{\theta^*}$ and θ^* assign the same vertex and edge labels to the trees GARLAND($\overline{\mathcal{S}}$) and GARLAND($\mathcal{S}$), respectively. By Proposition 3.1, $\overline{\theta^*}$ is a graceful labeling for the tree GARLAND($\overline{\mathcal{S}}$). Hence, θ^* is a graceful labeling for the tree GARLAND($\mathcal{S}$). ■

3.2 The Attachment Construction

Denote by ATTACHMENT($\mathcal{S}$) the tree constructed by unifying together the roots of all the trees in $\mathcal{S}$ into a single vertex r; we call r the *special vertex* of the constructed tree. The original construction is due to Koh *et al.* [9]; Goldenberg [5] calls it the *Attachment Construction*; a technical condition on the graceful labeling θ is assumed. Koh *et al.* [9] prove:

Proposition 3.3 *Consider a gracefully labeled tree $\langle T, \theta, w \rangle$ with $\theta(w) = |V(T)| - 1$. Let $\mathcal{S}$ consist of h copies of $\langle T, \theta, w \rangle$. Assume that*

$$\{\theta(u) \mid uw \in E(T)\} \quad \subset \quad \{0\} \cup \{(|V(T)| - 1) - \theta(u) \mid uw \in E(T)\}.$$

Then, the labeling

$$\theta^*(v) \quad = \quad \begin{cases} h(|V(T)| - 1)\,, & \text{if } v = r \\ \theta_i(v) + (h - i)(|V(T_i)| - 1)\,, & \text{if } v \in V(T_i) \setminus \{w_i\} \text{ and } \mathsf{dist}(v, w_i) \text{ is even} \\ \theta_i(v) + (i - 1)(|V(T_i)| - 1)\,, & \text{if } v \in V(T_i) \setminus \{w_i\} \text{ and } \mathsf{dist}(v, w_i) \text{ is odd} \end{cases}$$

is a graceful labeling for the tree ATTACHMENT($\mathcal{S}$).

Note that the relabeling function used on copy $\langle T_i, \theta_i, w_i \rangle$, with the single exception of its root w_i, is $\mathcal{R}^{\langle T_i, \theta_i, w_i \rangle}_{\langle 1, (h-i)(|V(T_i)|-1), (i-1)(|V(T_i)|-1) \rangle}$.

The Extended Attachment Construction requires that $\mathcal{S}$ be a gracefully consistent family; it returns a graceful labeling θ^* for the tree ATTACHMENT($\mathcal{S}$) as follows:

$$\theta^*(v) \quad = \quad \begin{cases} h(|V(T_1)| - 1) \quad (= h(|V(T_2)| - 1) = \ldots = h(|V(T_h)| - 1))\,, & \text{if } v = r \\ \mathcal{R}^{\langle T_i, \theta_i, w_i \rangle}_{\langle 1, (h-i)(|V(T_i)|-1), (i-1)(|V(T_i)|-1) \rangle}(v)\,, & \text{if } v \in V(T_i) \setminus \{w_i\} \end{cases}.$$

Figure 3 provides an illustration for the Extended Attachment Construction. We show:

Theorem 3.4 *Consider a gracefully consistent family $\mathcal{S}$ with $\theta_i(w_i) = |V(T_i)| - 1$, $1 \leq i \leq h$. Assume that*

$$\{\theta_h(u) \mid uw_h \in E(T_h)\} \quad \subset \quad \{0\} \cup \{(|V(T_h)| - 1) - \theta_h(u) \mid uw_h \in E(T_h)\}$$

and that $\{\theta_i(u) \mid uw_i \in E(T_i)\} = \{\theta_h(u) \mid uw_h \in E(T_h)\}$, for each i, $2 \leq i \leq h$. Then, the Extended Attachment Construction provides a graceful labeling θ^* for the tree ATTACHMENT($\mathcal{S}$).

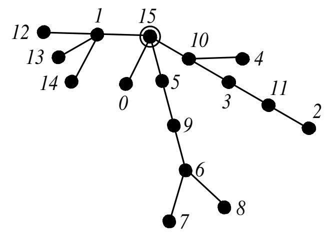


Figure 3: The tree ATTACHMENT($\mathcal{S}$), when $\mathcal{S}$ consists of the trees $\langle T_1, \theta_1, w_1 \rangle$, $\langle T_2, \theta_2, w_2 \rangle$, and $\langle T_3, \theta_3, w_3 \rangle$ taken from Figure 1. The special vertex of the constructed tree is circled.

Proof: Consider the family $\overline{\mathcal{S}} = \left\{ \langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle \mid \langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle \equiv \langle T_h, \theta_h, w_h \rangle,\ 1 \le i \le h \right\}$. Consider the following algorithm for labeling the tree ATTACHMENT($\overline{\mathcal{S}}$):

- For each $i,\ 1 \le i \le h$, do:
  - Label $\overline{r} \equiv \overline{w_i}$ with the value $|V(\overline{T_i})| - 1$.
  - Relabel the tree $\overline{T_i}$ using the function $\mathcal{R}^{\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle}_{\langle 1, (h-i)(|V(\overline{T_i})|-1), (i-1)(|V(\overline{T_i})|-1) \rangle}$.
- Label $\overline{r}$ with the value $h(|V(T_h)| - 1)$.

Clearly, the resulting labeling for the tree ATTACHMENT($\overline{\mathcal{S}}$) that is obtained by applying this algorithm is $\overline{\theta^*}$, which is a graceful labeling for the tree ATTACHMENT($\overline{\mathcal{S}}$) (by Proposition 3.3). Now consider the following algorithm for labeling the tree ATTACHMENT($\mathcal{S}$):

- For each $i,\ 1 \le i \le h$, do:
  - Label $r \equiv w_i$ with the value $|V(T_i)| - 1$.
  - Relabel the tree T_i using the function $\mathcal{R}^{\langle T_i, \theta_i, w_i \rangle}_{\langle 1, (h-i)(|V(T_i)|-1), (i-1)(|V(T_i)|-1) \rangle}$.
- Label r with the value $h(|V(T_h)| - 1)$.

Clearly, the resulting labeling for the tree ATTACHMENT($\mathcal{S}$) that is obtained by applying this algorithm is θ^*, which would also have been obtained if we had applied instead the Extended Attachment Construction on the tree ATTACHMENT($\mathcal{S}$). Therefore, to establish the claim it suffices to show that the two algorithms assign the same vertex and edge labels to each constituent tree.

Since $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle \equiv \langle T_h, \theta_h, w_h \rangle$ and $\langle T_i, \theta_i, w_i \rangle$ are gracefully consistent, $|V(\overline{T_i})| = |V(T_i)|$; thus, in the first step of the two algorithms above, for each $i,\ 1 \le i \le h$, $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle$ and $\langle T_i, \theta_i, w_i \rangle$ are relabeled using a relabeling function with the same triple of integers $\langle c, e, o \rangle$. Hence, by the Substitution Theorem, we get that for each $i,\ 1 \le i \le h$, immediately after the relabeling step in the i-th iteration: *(i)* the two algorithms assign the same labels to the roots of $\overline{T_i}$ and T_i, respectively; *(ii)* the two algorithms assign the same vertex and edge labels to the trees $\overline{T_i}$ and T_i, respectively.

We now examine two cases:

(1) Consider first the case $i = 1$. Following the first iteration of the two algorithms and the application of the same relabeling function to the trees $\overline{T_1}$ and T_1, their roots are both assigned the label $h(|V(\overline{T_1})| - 1) = h(|V(T_h)| - 1) = h(|V(T_1)| - 1)$; so, the two trees have the same vertex and edge labels. Their roots are their only vertices that change labels subsequently. Upon completion of executing the two algorithms, the roots are again assigned the label $h(|V(T_h)| - 1)$. So, each tree has the same vertex and edge labels as it had immediately following the first iteration; hence, the two trees have the same vertex and edge labels.

(2) Consider now the case $i > 1$. Following the relabeling step of the trees $\overline{T_i}$ and T_i, the only way the labels of those trees are affected by the execution of the two algorithms is by having the labels of their roots changed. Since both roots are eventually assigned the value $h(|V(T_h)|-1)$, it follows that at the end the two trees have the same vertex labels. It remains to show that they also have the same edge labels. The only edge labels that change are the labels of the edges that are adjacent to the tree roots. We show that these edge labels are affected in the same way in the trees $\overline{T_i}$ and T_i; thus, the two trees have the same edge labels at the end.

Let $\overline{N_i} = \{u \mid u\overline{w_i} \in E(\overline{T_i})\}$ (resp., $N_i = \{u \mid uw_i \in E(T_i)\}$), and let $\overline{L_{i,j}}$ (resp., $L_{i,j}$) denote the set of labels of vertices in $\overline{N_i}$ (resp., N_i), following iteration j of the first (resp., second) algorithm. When $j = i - 1$, the vertices of the trees $\overline{T_i}$ and T_i other than their roots are labeled according to the graceful labelings $\overline{\theta_i}$ and θ_i, respectively. Thus, $\overline{L_{i,i-1}} = \{\overline{\theta_i}(u) \mid u\overline{w_i} \in E(\overline{T_i})\}$ and $L_{i,i-1} = \{\theta_i(u) \mid uw_i \in E(T_i)\}$. By assumption, $\{\theta_i(u) \mid uw_i \in E(T_i)\} = \{\theta_h(u) \mid uw_h \in E(T_h)\}$; since $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle \equiv \langle T_h, \theta_h, w_h\rangle$, it also holds that $\{\overline{\theta_i}(u) \mid u\overline{w_i} \in E(\overline{T_i})\} = \{\theta_h(u) \mid uw_h \in E(T_h)\}$. Thus, $\overline{L_{i,i-1}} = L_{i,i-1}$.

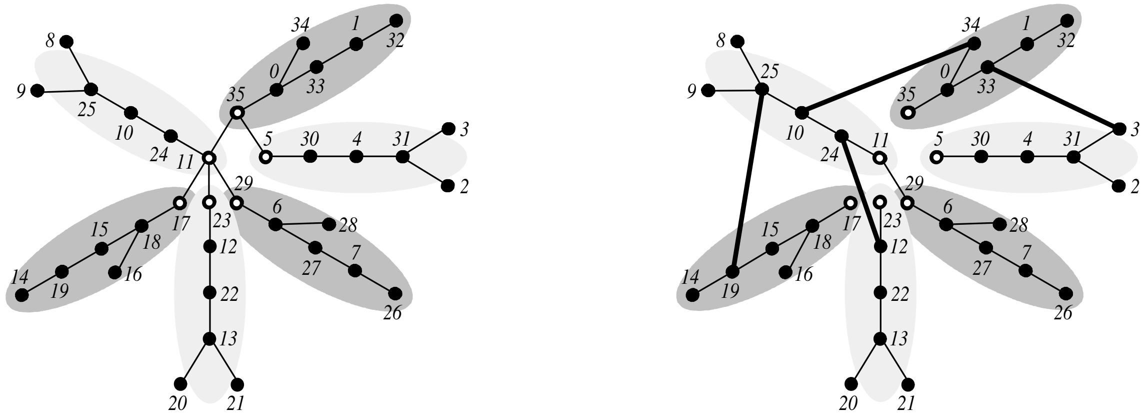
When $j = i$, the vertices in $\overline{N_i}$ and N_i are relabeled using the same relabeling function. Recall that vertices in $\overline{N_i}$ and N_i are all at odd distance from the roots of their respective trees; since $\overline{L_{i,i-1}} = L_{i,i-1}$, it follows by definition of relabeling function that $\overline{L_{i,i}} = L_{i,i}$. Since the two algorithms do not affect the labels of vertices in $\overline{N_i}$ and N_i during subsequent iterations, it holds that $\overline{L_{i,h}} = L_{i,h}$. Since the roots of $\overline{T_i}$ and T_i are assigned the same label after iteration h, the set of labels of the edges of the two trees that are adjacent to the roots are the same.

In conclusion, the two labelings $\overline{\theta^*}$ and θ^* assign the same vertex and edge labels to the trees ATTACHMENT($\overline{\mathcal{S}}$) and ATTACHMENT($\mathcal{S}$), respectively. By Proposition 3.3, $\overline{\theta^*}$ is a graceful labeling for the tree ATTACHMENT($\overline{\mathcal{S}}$). Hence, θ^* is a graceful labeling for the tree ATTACHMENT($\mathcal{S}$). ■

We note here that the two assumptions in the Extended Attachment Construction are a strict relaxation of the technical assumption of the original Attachment Construction, since now the special technical assumption need not hold for every tree. This is yet another generalization.

3.3 The Δ-Construction

Consider a gracefully labeled tree $\langle T_0, \theta_0, w_0\rangle$ with $V(T_0) = \{u_1, \ldots, u_h\}$, called the *host* tree. Denote by DELTA($\langle T_0, \theta_0, w_0\rangle, \mathcal{S}$) the tree constructed by unifying the root of every tree $\langle T_i, \theta_i, w_i\rangle$ in $\mathcal{S}$ with vertex u_i of the host tree $\langle T_0, \theta_0, w_0\rangle$. The original construction is due to Koh *et al.* [9], who call it the Δ-*Construction*. Koh *et al.* [9] prove:



(a) (b)

Figure 4: *(a)* The tree $\textsc{delta}(\langle T_1, \theta_1, w_1\rangle, \mathcal{S})$, when $\mathcal{S}$ consists of three copies of tree $\langle T_2, \theta_2, w_2\rangle$ and three copies of tree $\langle T_3, \theta_3, w_3\rangle$; the host and all constituent trees are taken from Figure 1. The dotted vertices are those of the host tree, which are identified with the roots of the constituent trees. *(b)* The gracefully labeled tree obtained by moving around some edges (drawn in bold) of the tree in Figure 4*(a)*.

Proposition 3.5 *Consider two gracefully labeled trees $\langle T, \theta, w\rangle$ and $\langle T_0, \theta_0, w_0\rangle$ with $V(T_0) = \{u_1, \ldots, u_h\}$. Let $\mathcal{S}$ consist of h copies of $\langle T, \theta, w\rangle$. Then, the labeling*

$$\theta^*(v) \quad = \quad \begin{cases} \theta_i(v) + \theta_0(u_i)|V(T_i)|\,, & \text{if } v \in V(T_i) \text{ and } \mathsf{dist}(v, w_i) \text{ is even} \\ \theta_i(v) + (h - \theta_0(u_i) - 1)|V(T_i)|\,, & \text{if } v \in V(T_i) \text{ and } \mathsf{dist}(v, w_i) \text{ is odd} \end{cases}$$

is a graceful labeling for the tree $\textsc{delta}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$.

Note that the relabeling function used on copy $\langle T_i, \theta_i, w_i\rangle$ is $\mathcal{R}^{\langle T_i,\theta_i,w_i\rangle}_{\langle 1,\theta_0(u_i)|V(T_i)|,(h-\theta_0(u_i)-1)|V(T_i)|\rangle}$. Note also that, unlike the Garland and Attachment Constructions, the Δ-Construction does not make any assumption on the roots of the trees in $\mathcal{S}$.

Burzio and Ferrarese [3] generalize the Δ-Construction by observing that for any two (identical) constituent trees $\langle T_i, \theta_i, w_i\rangle$ and $\langle T_j, \theta_j, w_j\rangle$ of the constructed tree $\textsc{delta}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$ such that T_i and T_j are attached to adjacent vertices u_i and u_j of the host tree, the edge $u_i u_j \equiv w_i w_j$ connecting their roots can be replaced by a new edge connecting two corresponding vertices in the identical trees T_i and T_j. Call the resulting construction the *Generalized Δ-Construction*.

The Extended Δ-Construction requires that $\mathcal{S}$ be a gracefully consistent family; it returns a graceful labeling θ^* for the tree $\textsc{delta}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$ as follows:

$$\theta^*(v) \quad = \quad \mathcal{R}^{\langle T_i,\theta_i,w_i\rangle}_{\langle 1,\theta_0(u_i)|V(T_i)|,(h-\theta_0(u_i)-1)|V(T_i)|\rangle}(v)\,, \quad \text{if } v \in V(T_i) \quad .$$

Figure 4*(a)* provides an illustration for the Extended Δ-Construction. We show:

Theorem 3.6 *Consider a gracefully consistent family $\mathcal{S}$ and a gracefully labeled tree $\langle T_0, \theta_0, w_0\rangle$ with $V(T_0) = \{u_1, \ldots, u_h\}$. Then, the Extended Δ-Construction provides a graceful labeling θ^* for the tree* $\textsc{delta}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$.

Proof: Consider the family $\overline{\mathcal{S}} = \left\{ \langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle \mid \langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle \equiv \langle T_h, \theta_h, w_h \rangle,\ 1 \leq i \leq h \right\}$. By Proposition 3.5, the tree $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \overline{\mathcal{S}})$ is graceful. Recall that the labeling $\overline{\theta^*}$ for the tree $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \overline{\mathcal{S}})$ is obtained by relabeling every tree $\overline{T_i}$, $1 \leq i \leq h$ using the function $\mathcal{R}^{\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle}_{\langle 1, \theta_0(u_i)|V(\overline{T_i})|, (h-\theta_0(u_i)-1)|V(\overline{T_i})| \rangle}$; the labeling θ^* for the tree $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \mathcal{S})$ is obtained by relabeling every tree T_i, $1 \leq i \leq h$ using the function $\mathcal{R}^{\langle T_i, \theta_i, w_i \rangle}_{\langle 1, \theta_0(u_i)|V(T_i)|, (h-\theta_0(u_i)-1)|V(T_i)| \rangle}$.

Since $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle \equiv \langle T_h, \theta_h, w_h \rangle$ and $\langle T_i, \theta_i, w_i \rangle$ are gracefully consistent, $|V(\overline{T_i})| = |V(T_i)|$; thus, for each i, $1 \leq i \leq h$, $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i} \rangle$ and $\langle T_i, \theta_i, w_i \rangle$ are relabeled using a relabeling function with the same triple of integers $\langle c, e, o \rangle$. Hence, by the Substitution Theorem, we get that for each i, $1 \leq i \leq h$: *(i)* $\overline{\theta^*}$ and θ^* assign the same labels to the roots of $\overline{T_i}$ and T_i, respectively; *(ii)* $\overline{\theta^*}$ and θ^* assign the same vertex and edge labels to the trees $\overline{T_i}$ and T_i, respectively. So, $\overline{\theta^*}$ and θ^* assign the same edge labels to the edges that connect the roots of the constituent trees in $\overline{\mathcal{S}}$ and $\mathcal{S}$, respectively; these are the edges of the host trees.

In conclusion, the two labelings $\overline{\theta^*}$ and θ^* assign the same vertex and edge labels to the trees $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \overline{\mathcal{S}})$ and $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \mathcal{S})$, respectively. By Proposition 3.5, $\overline{\theta^*}$ is a graceful labeling for the tree $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \overline{\mathcal{S}})$. Hence, θ^* is a graceful labeling for the tree $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \mathcal{S})$. ■

We generalize the Extended Δ-Construction to the *Generalized Extended Δ-Construction*, much in the same way that the Δ-Construction is generalized to the Generalized Δ-Construction by Burzio and Ferrarese [3]: Consider any two constituent trees $\langle T_i, \theta_i, w_i \rangle$ and $\langle T_j, \theta_j, w_j \rangle$ of the tree $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \mathcal{S})$ such that T_i and T_j are attached to adjacent vertices u_i and u_j of the host tree. Replace the edge $u_i u_j \equiv w_i w_j$ with an edge $v_i v_j$ where v_i and v_j are vertices of the trees T_i and T_j, respectively, that had been assigned the same label in the original graceful labelings θ_i and θ_j, respectively. Figure 4*(b)* provides an illustration for the Generalized Extended Δ-Construction. We show:

Theorem 3.7 *Consider a gracefully consistent family $\mathcal{S}$ and a gracefully labeled tree $\langle T_0, \theta_0, w_0 \rangle$ with $V(T_0) = \{u_1, \ldots, u_h\}$. Then, the Generalized Extended Δ-Construction provides a graceful labeling θ^* for the tree obtained by moving around some edges of the tree $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \mathcal{S})$ as described in the preceding paragraph.*

Proof: Let θ^* be the graceful labeling for the tree $\text{DELTA}(\langle T_0, \theta_0, w_0 \rangle, \mathcal{S})$, whose gracefulness is established by Theorem 3.6. Recall that the distances $\mathsf{dist}(v_i, w_i)$ and $\mathsf{dist}(v_j, w_j)$ of the vertices v_i and v_j from the roots w_i and w_j, respectively, are either both even or both odd. By the Extended Δ-Construction, it follows that either $\theta^*(v_i) = \theta_i(v_i) + \theta_0(u_i)|V(T_i)|$ and $\theta^*(v_j) = \theta_j(v_j) + \theta_0(u_j)|V(T_j)|$, or $\theta^*(v_i) = \theta_i(v_i) + (h - \theta_0(u_i) - 1)|V(T_i)|$ and $\theta^*(v_j) = \theta_j(v_j) + (h - \theta_0(u_j) - 1)|V(T_j)|$. Since $\langle T_i, \theta_i, w_i \rangle$ and $\langle T_j, \theta_j, w_j \rangle$ are gracefully consistent, $|V(T_i)| = |V(T_j)|$. Since also $\theta_i(v_i) = \theta_j(v_j)$, it follows that in all cases, $|\theta^*(v_i) - \theta^*(v_j)| = |\theta_0(u_i) - \theta_0(u_j)||V(T_i)|$. By the Extended Δ-Construction, $\theta^*(w_i) = \theta_i(w_i) + \theta_0(u_i)|V(T_i)|$ and $\theta^*(w_j) = \theta_j(w_j) + \theta_0(u_j)|V(T_j)|$. Since $\langle T_i, \theta_i, w_i \rangle$ and $\langle T_j, \theta_j, w_j \rangle$ are gracefully consistent, $|V(T_i)| = |V(T_j)|$ and $\theta_i(w_i) = \theta_j(w_j)$. It follows that $|\theta^*(v_i) - \theta^*(v_j)| = |\theta^*(w_i) - \theta^*(w_j)|$, so that the label of the removed edge $u_i u_j \equiv w_i w_j$ is the same as the label of the added edge $v_i v_j$. Hence, θ^* is a graceful labeling for the tree constructed under the Generalized Extended Δ-Construction. ■

3.4 The Δ_{+1}-Construction

Consider a gracefully labeled tree $\langle T_0, \theta_0, w_0\rangle$ with $V(T_0) = \{u_1, \ldots, u_h, w_0\}$, called the *host* tree. Denote by $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$ the tree constructed by the following procedure:

- Remove the root w_0 of $\langle T_0, \theta_0, w_0\rangle$ and all its adjacent edges.
- Unify the root of every tree $\langle T_i, \theta_i, w_i\rangle$ in $\mathcal{S}$ with vertex u_i of the host tree $\langle T_0, \theta_0, w_0\rangle$.
- Add a distinguished vertex r; we call r the *special vertex* of the constructed tree.
- For every tree $\langle T_i, \theta_i, w_i\rangle$ in $\mathcal{S}$ such that $u_i w_0$ is an edge of T_0, connect r to 0_{θ_i}.

The original construction is due to Burzio and Ferrarese [3], who call it the Δ_{+1}-*Construction*; a condition on the 0-vertex of T is assumed. Burzio and Ferrarese [3] prove:

Proposition 3.8 *Consider two gracefully labeled trees $\langle T, \theta, w\rangle$ and $\langle T_0, \theta_0, w_0\rangle$ with $V(T_0) = \{u_1, \ldots, u_h, w_0\}$ and $\theta_0(w_0) = h$. Let $\mathcal{S}$ consist of h copies of $\langle T, \theta, w\rangle$. Assume that the 0-vertex of T is at even distance from its root w. Then, the labeling*

$$\theta^*(v) \quad = \quad \begin{cases} h|V(T)|\,, & \textit{if } v = r \\ \theta_i(v) + \theta_0(u_i)|V(T_i)|\,, & \textit{if } v \in V(T_i) \textit{ and } \mathsf{dist}(v, w_i) \textit{ is even} \\ \theta_i(v) + (h - \theta_0(u_i) - 1)|V(T_i)|\,, & \textit{if } v \in V(T_i) \textit{ and } \mathsf{dist}(v, w_i) \textit{ is odd} \end{cases}$$

is a graceful labeling for the tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$.

Note that the relabeling function used on copy $\langle T_i, \theta_i, w_i\rangle$ is $\mathcal{R}^{\langle T_i, \theta_i, w_i\rangle}_{\langle 1, \theta_0(u_i)|V(T_i)|, (h-\theta_0(u_i)-1)|V(T_i)|\rangle}$. Note also that, unlike the Garland and Attachment Constructions, the Δ_{+1}-Construction only makes an indirect (and weaker) assumption on the roots of the trees in $\mathcal{S}$.

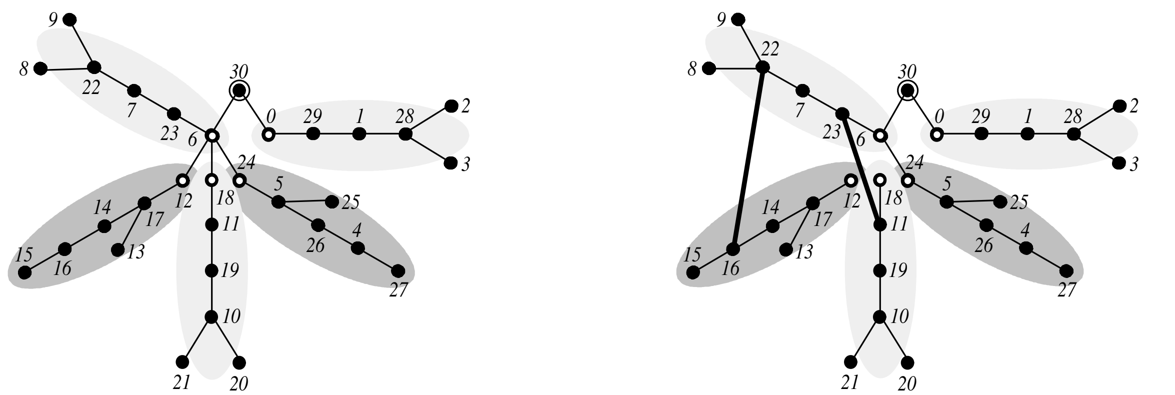
We generalize the Δ_{+1}-Construction by observing that for any two (identical) constituent trees $\langle T_i, \theta_i, w_i\rangle$ and $\langle T_j, \theta_j, w_j\rangle$ of the constructed tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$ such that T_i and T_j are attached to adjacent vertices u_i and u_j of the host tree, the edge $u_i u_j \equiv w_i w_j$ connecting their roots can be replaced by a new edge connecting two corresponding vertices in the identical trees T_i and T_j. Call the resulting construction the *Generalized* Δ_{+1}-*Construction.*

The Extended Δ_{+1}-Construction requires that $\mathcal{S}$ be a gracefully consistent family; it returns a graceful labeling θ^* for the tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$ as follows:

$$\theta^*(v) \quad = \quad \begin{cases} h|V(T_1)| \quad (= h|V(T_2)| = \ldots = h|V(T_h)|)\,, & \text{if } v = r \\ \mathcal{R}^{\langle T_i, \theta_i, w_i\rangle}_{\langle 1, \theta_0(u_i)|V(T_i)|, (h-\theta_0(u_i)-1)|V(T_i)|\rangle}(v)\,, & \text{if } v \in V(T_i) \end{cases} .$$

Figure 5*(a)* provides an illustration for the Extended Δ_{+1}-Construction. We show:

Theorem 3.9 *Consider a gracefully consistent family $\mathcal{S}$ and a gracefully labeled tree $\langle T_0, \theta_0, w_0\rangle$ with $V(T_0) = \{u_1, \ldots, u_h, w_0\}$ and $\theta_0(w_0) = h$. Assume that the 0-vertex of T_i is at even distance from its root w_i, for every $\langle T_i, \theta_i, w_i\rangle \in \mathcal{S}$ such that $u_i w_0 \in E(T_0)$. Then, the Extended Δ_{+1}-Construction provides a graceful labeling θ^* for the tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$.*



(a) (b)

Figure 5: *(a)* The tree $\text{DELTA}_{+1}(\langle T_1, \theta_1, w_1\rangle, \mathcal{S})$, when $\mathcal{S}$ consists of three copies of tree $\langle T_2, \theta_2, w_2\rangle$ and two copies of tree $\langle T_3, \theta_3, w_3\rangle$; the host and all constituent trees are taken from Figure 1. The dotted vertices are those of the host tree other than its root, which are identified with the roots of the constituent trees; the special vertex of the constructed tree is circled. *(b)* The gracefully labeled tree obtained by moving around some edges (drawn in bold) of the tree in Figure 5*(a)*.

Proof: Consider the family $\overline{\mathcal{S}} = \left\{\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle \mid \langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle \equiv \langle T_h, \theta_h, w_h\rangle,\ 1 \le i \le h\right\}$. By Proposition 3.8, the tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \overline{\mathcal{S}})$ is graceful. Recall that the labeling $\overline{\theta^*}$ for the tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \overline{\mathcal{S}})$ is obtained by relabeling every tree $\overline{T_i},\ 1 \le i \le h$ using the function $\mathcal{R}^{\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle}_{\langle 1, \theta_0(u_i)|V(\overline{T_i})|, (h-\theta_0(u_i)-1)|V(\overline{T_i})|\rangle}$; the labeling θ^* for the tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$ is obtained by relabeling every tree $T_i,\ 1 \le i \le h$ using the function $\mathcal{R}^{\langle T_i, \theta_i, w_i\rangle}_{\langle 1, \theta_0(u_i)|V(T_i)|, (h-\theta_0(u_i)-1)|V(T_i)|\rangle}$.

Since $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle \equiv \langle T_h, \theta_h, w_h\rangle$ and $\langle T_i, \theta_i, w_i\rangle$ are gracefully consistent, $|V(\overline{T_i})| = |V(T_i)|$; thus, for each $i,\ 1 \le i \le h$, the trees $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle$ and $\langle T_i, \theta_i, w_i\rangle$ are relabeled using a relabeling function with the same triple of integers $\langle c, e, o\rangle$. Hence, by the Substitution Theorem, we get that for each $i,\ 1 \le i \le h$: *(i)* $\overline{\theta^*}$ and θ^* assign the same labels to the roots of $\overline{T_i}$ and T_i, respectively; *(ii)* $\overline{\theta^*}$ and θ^* assign the same vertex and edge labels to the trees $\overline{T_i}$ and T_i, respectively. So, $\overline{\theta^*}$ and θ^* assign the same edge labels to the edges that connect the roots of the constituent trees in $\overline{\mathcal{S}}$ and $\mathcal{S}$, respectively; these are the edges of the host trees.

Recall that the final step of the Extended Δ_{+1}-Construction connects the 0-vertex of every constituent tree $\langle \overline{T_i}, \overline{\theta_i}, \overline{w_i}\rangle$ in $\overline{\mathcal{S}}$ such that $u_i w_0 \in E(T_0)$ to the special vertex $\overline{r}$ of the constructed tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \overline{\mathcal{S}})$; analogously, it connects the 0-vertex of every constituent tree $\langle T_i, \theta_i, w_i\rangle$ in $\mathcal{S}$ such that $u_i w_0 \in E(T_0)$ to the special vertex r of the constructed tree $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$. By assumption, the 0-vertices that are connected to $\overline{r}$ and r are at even distances from the original roots of their respective constituent trees. Hence, by definition of relabeling function, they are relabeled in the same way, and end up, therefore, with the same label. Since $\overline{r}$ and r are assigned the same label in the trees $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \overline{\mathcal{S}})$ and $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$, respectively, it follows that the edges connecting the 0-vertices of the constituent trees to $\overline{r}$ and r, respectively, are also assigned the same edge labels. Thus, the last step in the Extended Δ_{+1}-Construction preserves the labels between the trees $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \overline{\mathcal{S}})$ and $\text{DELTA}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$.

In conclusion, the two labelings $\overline{\theta^*}$ and θ^* assign the same vertex and edge labels to the trees

$\textsc{delta}_{+1}(\langle T_0, \theta_0, w_0\rangle, \overline{\mathcal{S}})$ and $\textsc{delta}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$, respectively. By Proposition 3.8, $\overline{\theta^*}$ is a graceful labeling for the tree $\textsc{delta}_{+1}(\langle T_0, \theta_0, w_0\rangle, \overline{\mathcal{S}})$. Hence, θ^* is a graceful labeling for the tree $\textsc{delta}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$. ∎

We note here that the assumption on the Extended Δ_{+1}-Construction is a strict relaxation of the technical assumption of the original Δ_{+1}-Construction, since now the special technical assumption need not hold for every tree. This is yet another generalization.

We generalize the Extended Δ_{+1}-Construction to the *Generalized Extended* Δ_{+1}*-Construction*, much in the same way that the Δ_{+1}-Construction is generalized to the Generalized Δ_{+1}-Construction: Consider any two constituent trees $\langle T_i, \theta_i, w_i\rangle$ and $\langle T_j, \theta_j, w_j\rangle$ of the tree $\textsc{delta}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$ such that T_i and T_j are attached to adjacent vertices u_i and u_j of the host tree. Replace the edge $u_iu_j \equiv w_iw_j$ with an edge v_iv_j where v_i and v_j are vertices of the trees T_i and T_j, respectively, that had been assigned the same label in the original graceful labelings θ_i and θ_j, respectively. Figure 5*(b)* provides an illustration for the Generalized Extended Δ_{+1}-Construction. We show:

Theorem 3.10 *Consider a gracefully consistent family $\mathcal{S}$ and a gracefully labeled tree $\langle T_0, \theta_0, w_0\rangle$ with $V(T_0) = \{u_1, \ldots, u_h, w_0\}$ and $\theta_0(w_0) = h$. Assume that the 0-vertex of T_i is at even distance from its root w_i, for every $\langle T_i, \theta_i, w_i\rangle \in \mathcal{S}$ such that $u_iw_0 \in E(T_0)$. Then, the Generalized Extended Δ_{+1}-Construction provides a graceful labeling θ^* for the tree obtained by moving around some edges of the tree $\textsc{delta}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$ as described in the preceding paragraph.*

Proof: Let θ^* be the graceful labeling for the tree $\textsc{delta}_{+1}(\langle T_0, \theta_0, w_0\rangle, \mathcal{S})$, whose gracefulness is established by Theorem 3.9. Recall that the distances $\mathsf{dist}(v_i, w_i)$ and $\mathsf{dist}(v_j, w_j)$ of the vertices v_i and v_j from the roots w_i and w_j, respectively, are either both even or both odd. By the Extended Δ_{+1}-Construction, it follows that either $\theta^*(v_i) = \theta_i(v_i) + \theta_0(u_i)|V(T_i)|$ and $\theta^*(v_j) = \theta_j(v_j) + \theta_0(u_j)|V(T_j)|$, or $\theta^*(v_i) = \theta_i(v_i) + (h - \theta_0(u_i) - 1)|V(T_i)|$ and $\theta^*(v_j) = \theta_j(v_j) + (h - \theta_0(u_j) - 1)|V(T_j)|$. Since $\langle T_i, \theta_i, w_i\rangle$ and $\langle T_j, \theta_j, w_j\rangle$ are gracefully consistent, $|V(T_i)| = |V(T_j)|$. Since also $\theta_i(v_i) = \theta_j(v_j)$, it follows that in all cases, $|\theta^*(v_i) - \theta^*(v_j)| = |\theta_0(u_i) - \theta_0(u_j)||V(T_i)|$. By the Extended Δ_{+1}-Construction, $\theta^*(w_i) = \theta_i(w_i) + \theta_0(u_i)|V(T_i)|$ and $\theta^*(w_j) = \theta_j(w_j) + \theta_0(u_j)|V(T_j)|$. Since $\langle T_i, \theta_i, w_i\rangle$ and $\langle T_j, \theta_j, w_j\rangle$ are gracefully consistent, $|V(T_i)| = |V(T_j)|$ and $\theta_i(w_i) = \theta_j(w_j)$. It follows that $|\theta^*(v_i) - \theta^*(v_j)| = |\theta^*(w_i) - \theta^*(w_j)|$, so that the label of the removed edge $u_iu_j \equiv w_iw_j$ is the same as the label of the added edge v_iv_j. Hence, θ^* is a graceful labeling for the tree constructed under the Generalized Extended Δ_{+1}-Construction. ∎

4 Conclusion

We presented a Substitution Theorem for graceful trees as a combinatorial tool for the enlargement of known graceful classes of trees. In turn, we applied the Substitution Theorem on several known constructions [3, 7, 9]. Our results extend the class of trees known to be graceful. Wu [16, 17] has recently and independently investigated alternative extensions for the particular case of the Garland Construction [7] to families of bipartite or isomorphic graphs.

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